A Path Through the Wilderness: Time Discounting in Growth Models[1]

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Introduction

Time is a pervasive element of many economic activities. As Jevons (1888, 64-65) once said:

> Quantity of supply must necessarily be estimated by the number of units of commodity divided by the number of units in the time over which it is to be expended. … Consumption of commodity must have the same dimensions. For goods must be consumed in time… .
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> Following out this course of thought we shall arrive at the conclusion that time enters into all economic questions. We live in time, and think and act in time; we are in fact altogether the creatures of time.

Even though economic decision-making can be seen as an intrinsically intertemporal process, the ways economists treated and modeled time has varied substantially in the last century. More specifically, from the 1930s to the postwar period, economists discussing issues of economic growth changed in important ways their modeling strategies with respect to discounting the future. In Cambridge in the 1930s, economists like Arthur Cecil Pigou (1912, [1920] 1924) and Frank Ramsey (1928) condemned the use of a discount factor to the utility of future generations. However, for postwar growth economists, who consider Ramsey to be one of their patron saints, the discount factor, applied either to individual's or to social planner's decision making, is a technical requirement of dynamic general equilibrium models.[2]

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[1] I am grateful to participants at the HOPE lunch group at Duke University for comments and suggestions when the ideas I explore here were at an incipient stage. I thank also participants at the 2009 HISRECO (Antwerp, Belgium). This is a preliminary and incomplete text based on a work in progress for presentation at the HES 2009 (Denver). I thank FAPESP (Brazil) for financial support. Please, do not quote without permission. Comments are very welcome.

[2] In most present-day graduate macroeconomics textbooks the discount factor is treated as a technically indispensable hypothesis not only of growth models but also of dynamic general equilibrium models in general: Blanchard and Fischer (1989), Stokey and Lucas (1989), Obstfeld and Rogoff (1996), Barro and Sala-i-Martin (2003), Woodford (2003), and Ljungqvist and Sargent (2004).

In present-day macroeconomics, discounting future utilities at a constant rate is one way of assuring the intertemporal maximization problem is well defined. Therefore, economists can rely on the so-called contraction-mapping theorem that gives an easy recursive solution to their problems (Stokey and Lucas 1989). On the other hand, the idea that consumers are impatient with respect to future utilities became a common interpretation of these models either when applied to a representative agent's or to a social planner's maximization problem.

The stabilization of time discounting in postwar economics serves as a case study to understand better different aspects of the mathematization and the dominance of the neoclassicism in modern economics (Morgan and Rutherford 1998, Mirowski 2002, and Weintraub 1991, 2002). My goal is to analyze this episode in more details, by looking at how economists working with what we now call economic growth models, operating in particular communities, assumed that individuals and generations discount future exponentially with a constant discount rate.

One way of starting this narrative is to look at the early uses of time discounting in the literature on economic dynamics in the period from the 1920s to the 1950s. Although for the most part these models did not discuss economic growth, they do illustrate of how economists and mathematicians then working with formal intertemporal models and using calculus of variations had no common understanding about when to discount the future. This does not mean that they were ignorant of ideas that goods or payments differed in time are in fact different things from today's perspective. To the contrary, despite of knowing this they did or did not use time discounting in their models depending on the problems they tackled.

In the 1950s and 1960s the burgeoning literature on economic growth discussed more explicitly the convergence conditions that guarantee the existence of solutions to intertemporal models. Nonetheless, different notions of convergence (with the use of a discount factor being just one among them) were used at that time and it was even possible to see economists presenting their models with and without time discounting—as did, for instance, Edmund Phelps (1967) in discussing welfare-maximizing economic policies by borrowing some ideas from Ramsey (1928) and the growth literature. Roughly from the 1970s macroeconomists treated time discounting more and more as a technical requirement of their models.

In pondering the different moves of this narrative, there are broad changes occurring in economics that are relevant to it. The fact that time discounting became a technical requirement in economics relates to the dominance of formal Walrasian general-equilibrium models in the last half of the last century.[3] A crucial element here is the axiomatization of time preferences that economists like Tjalling Koopmans worked out: it transformed the concept of impatience previously introduced by Irving Fisher (1930, chap. 4) into the element defining an ordinal utility function; and such function in turn defines time preferences either of an individual consumer, or of an aggregate of such individuals, or finally choices in a centrally planned economy (Koopmans 1960, 288). Add to this the increasing use of a representative agent in macroeconomic models in general, and in growth models in particular, and you get the overall picture of the current understanding that a time discount factor is a technical requirement of either a decentralized or a social planner's problem.

Moreover, the use of time discounting as a technical component of growth models goes hand in hand with the emergence of the so-called "Ramsey-Cass-Koopmans model," in which a representative agent discounts the future at a constant rate (Duarte forthcoming (b)).[4] In this respect it is also important to note the changes in method that happened in the growth literature with the Hamiltonian formalism (Wulwick 1995).

As one sees the narrative of the use of a time discount rate in growth models is no straightforward account. My goal in this article is to highlight some major elements that help us understand how a practice that was condemned as ethically indefensible when applied to intergenerational comparisons became a technical requirement in dynamic models of either a consumer or a planner deciding the intertemporal allocation of resources.

1. An ethically indefensible practice

In the first half of the twentieth century there was in Cambridge a tradition against discounting utilities of future generations. For instance, Arthur Cecil Pigou (1912, [1920] 1924) argued, following Sidgwick's utilitarianism, that "there was no philosophical basis for not treating future people just like present people. But in practice, because of a 'defective telescopic faculty' and the vagaries of time and blood … future people would actually count for less" (Collard 1996, 587). Pigou argued that because the present generations have a defective telescopic faculty—their members see things distant in time as being smaller than current ones—and also as a consequence of mortality and weak linkages over time, they invest relatively few resources today, especially in

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[3] See De Vroey (2004) for a history of the macroeconomics after Keynes as a general movement from Marshallian to Walrasian general equilibrium models.

[4] See also other papers of the HOPE 2008 conference for an analysis of other aspects of the development of the growth literature in the 1960s.

human capital. Therefore, according to Pigou, the government could intervene to guarantee a fairer generational justice. Collard (1996) analyzed in details this issue and showed that there was "a strong Cambridge tradition (Mill-Sidgwick-Marshall-Pigou-Ramsey) against discounting future utilities" (585). For Pigou discounting future utilities was mistaken because it led to sub-optimal intertemporal allocation of consumption.

Frank P. Ramsey (1928), the Cambridge mathematician who had Pigou as a mentor and supporter in economics (Duarte forthcoming (a)), took a stronger stand than the latter and stated that the use of a discount rate when one discusses the intertemporal allocation of resources by the society as a whole is ethically indefensible because generations ought to be treated equally. In this context he "assumed that we do not discount later enjoyments in comparison to earlier ones, a practice which is ethically indefensible and arises merely from weakness of the imagination" (Ramsey 1928, 543).

Then how did Ramsey guarantee that his intertemporal maximization problem would have a solution, when no discount rate is used and, thus, the integral of current and future utilities may be improper? He used a trick that became well known after the publication of his 1928 paper: he assumed that the total utility of consumption net of the disutility of labor has an upper bound, given either by satiation of utility or of technological restrictions (so that a greater stock of capital would increase neither income nor leisure), which he called "Bliss." He then represented his intertemporal maximization problem (in continuous time) as the minimization of the integral, from the present to the indefinite future, of the deviations of current net utility from the bliss level:

$$\underset{x_t, a_t}{Min} \int_{t=0}^{\infty} \{B - [U(x_t) - V(a_t)]\} \cdot dt \qquad s.t.: \ \frac{dc_t}{dt} + x_t = f(a_t, c_t)$$

where B is the bliss level; x_t, a_t, and c_t denote respectively consumption, labor and the stock of capital; $U(x_t) - V(a_t)$ is the "net enjoyment per unit of time" (the difference between the utility of consumption, $U(x_t)$, and the disutility of labor, $V(a_t)$); $f(x_t, a_t)$ is the production function; and the constraint to this minimization problem is the economy's resource constraint that states that investment plus consumption should equal total output.

It should be stressed here that despite being a mathematical trick to guarantee that such integral does converge (and thus he could characterize the properties of the solution of the intertemporal problem), Ramsey did present economic arguments to justify such this hypothesis. He stated that the net utility that solves the above problem is an increasing function of the capital stock "since with more capital we can obtain more enjoyment" (544). However, this utility ceases to increase for either of two reasons: (1) from the production side of the economy, further increases in capital "would not enable us to increase either our income or our leisure"; (2) from the demand side of the economy (consumers), the net utility reaches a maximum conceivable level. In either case, there exists a finite level of capital that is associated with the maximum rate of net utility economically obtainable (whether or not it corresponds to the maximum conceivable).

Ramsey (1928) then considered a more general case: one in which the net utility always increases with increments to the capital stock. In this general case there are two sub-cases that he considered: first that utility approaches asymptotically a certain finite limit (a finite supremum, in mathematical terms—not used by Ramsey in the article), and second that it increases without bound. Clearly, if the latter was possible, Ramsey could not go on with a notion like "bliss." He dismissed this mathematical possibility with a brief economic argument: "we shall dismiss [the possibility of the net enjoyment increasing to infinity] on the ground that economic causes alone could never give us more than a certain finite rate of enjoyment (called … the maximum conceivable rate)" (545).

Besides discussing in economic terms his mathematical approach, Ramsey (1928) did extend his results and analysis to the case of one individual or community that discount future utilities at a constant rate, despite warning against the use of such discounting in an intergenerational problem (whose solution had to rest on the existence of a bliss level). We see Ramsey using time discounting both in the published article and the notes in which he not only sketched the mathematical calculations for this article but also analyzed the case of taxing savings (Ramsey forthcoming).[5] The reason for using a discount rate is that Ramsey (1928) implicitly considers his approach applicable either to an individual or to a "community that goes on forever without changing either in numbers or in its capacity of enjoyment or in its aversion to labour" (543).[6] In going from one case to another, from the individual to the community, Ramsey is not

[5] Keynes asked Ramsey to cut this section on taxation from the article published in 1928 in the *Economic Journal*, as discussed by Duarte (Forthcoming (a)).

[6] In the first page of the article, immediately after stating that discounting the utility of future generations is ethically indefensible, Ramsey (1928, 543) says that he "shall … in Section II include such a rate of discount in some of our investigations." But even in the first section, where he analyzes the case of a community, after presenting Keynes's intuition for his main result he mentions that the set of equations derived can be extended to the case of discounting future utilities while Keynes's intuition cannot—thus, discounting could be used to the case of a community. Then, in section II, when extending his previous analysis to the discounting case, Ramsey (1928, 553) is only concerned with not contradicting his initial hypothesis that "successive generations are actuated by the same system of preferences." For this, he assumed that the rate of discount is constant, but this does not mean that "it is the same for all individuals,

explicit about whose utility function he analyzed, whether of a social planner (community) or of one individual who would represent the utility of all others: as explored in Duarte (2008), there is some evidence that Ramsey may have had in mind a notion of a representative agent similar to that widely used in macroeconomics nowadays. If this is indeed the case, his criticisms to using a time discount factor to the utilities of future generations may disappear in a representative agent world because generations do not differ in any substantial manner (i.e., in their utility of consumption and disutility of labor).

2. Time discounting in economic dynamics, 1920s-1940s

In this section I want to show a brief panorama of uses of a discount rate by mathematicians and economists working on economic dynamics from the 1920s to the 1940s (with more emphasis on the 1930s and the 1940s). The goal is to look at a literature similar to the optimal growth literature that emerged in the 1950s: one in which there is an explicit intertemporal optimization problem solved with calculus of variations and in which time discounting may or may not be employed. Therefore, it is a branch of economic dynamics related to names like the mathematicians Griffith Evans (1887-1973) and his advisee Charles F. Roos (1901-1958), one of the founders of the Econometric Society in 1930, and the economists Albert Gailord Hart (1909-1997) and Gerhard Tintner (1907-1983)—not to Paul Samuelson.[7]

The dynamics expounded by Samuelson ([1947] 1983), and reaffirmed by him in a survey of this field for the American Economic Association (Samuelson 1948), is that of casting economic phenomena in terms either of difference or of differential equations and then analyzing the dynamics that emerge after imposing restrictions (generally in terms of parameters) that make comparative-static exercises render sensible results, given initial conditions. Here, as in the dynamics of Evans, Roos and the others, "there are functional relationships between economic variables and their rates of change, their 'velocities,' 'accelerations,' or higher 'derivatives of derivatives'" (Samuelson 1948, 354), but there is no explicit optimization problems to be solved. Therefore, Samuelson did not analyze the issue of time discounting in this context notwithstanding his earlier use of discounting when discussing either the measurement of utility or capital theory (Samuelson 1937(a), (b)).

since we are at present only concerned with one individual or community." In the set of notes published (Ramsey forthcoming), we see that he struggled with the case of a time varying rate of discount.

[7] Both Roos and Tintner were members of the Cowles Commission in the 1930s: Roos helped found Cowles and was a research director from 1934 to 1937; Tintner was a research fellow for a bit less than a year, 1936-1937.

2.1. The mathematicians: Evans and Roos

The main feature of Griffith Evans's and Charles Roos's research agendas that is of interest here is that they introduced the rate of price changes as an argument of the demand function.[8] As a consequence, they turned the problems of competition and monopoly into problems in the calculus of variations (Roos 1927, 635) in which firms choose prices that maximize profits over an interval of time, subject to a demand function (time is continuous in their models). It is in expressing this integral of profits that they do or do not take into account a discount factor.

Evans (1924, 1925, 1930 chaps. 14-15) considers for simplicity, and avoiding further behavioral assumptions,[9] the case of a quadratic cost function and basically a demand function that is a linear differential equation without having an explicit term involving time.[10] He also mentions the cases of a linear demand function with a time argument: the case suggested by Irving Fisher according to which the volume of traded goods at time t depends on the rate of change of the price index at a previous time, and that of a law of demand—called "integral demand law" (Evans 1930, 46) —in which the quantity demanded depends on a weighted average of either past prices (Evans 1925, 108) or past quantities (Evans 1930, 158-62)—with a weighting function that decreases with the time elapsed from the past to the present.[11] In all cases, Evans assumed that the monopolistic firm chooses a function $p(t)$ that maximizes the undiscounted integral of profits over the interval (t_1, t_2) (subject to a demand function):

$$\underset{p(t)}{Max} \quad \Pi = \int_{t_1}^{t_2} \pi(p, p', t) \cdot dt$$

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[8] As Roos (1927, 635) points out, "a number of the Department of Agriculture economists have been" introducing the time element explicitly into the demand functions "in the last few years."

[9] Weintraub 2002, chap. 2, argued that Evans, "reprising Volterra's critique of Pareto more than a quarter century earlier" understood that "in mathematical economics one should not be so concerned with the behavioral theories themselves" (69). Economic theory should derive implications "testable either empirically through data analysis or through common sense" (70).

[10] Griffith Evans earned his Ph.D. at Harvard in 1910 and subsequently was a postdoctoral student of Vito Volterra at the University of Rome, which "was to be the marker event of his intellectual life" (Weintraub 2002, 42). From 1912 to 1933 Evans was a teacher at the Rice Institute (nowadays Rice University). His papers published in the 1920s in mathematical journals, which led up to his 1930 book, all "called attention of mathematicians to interesting problems in an applied discipline" (Weintraub 2002, 57).

[11] Evans (1930, 158 n. 1) explains that Volterra was the first to discuss this type of law in problems in physics and that Roos, his student at Rice, applied it to the theory of competition in his MA thesis and in his 1925 article. Roos obtained a BA degree in 1921, received his MA in 1924 and his Ph.D. in 1926, all at Rice University. From 1926 to 1928 he was a National Research Fellow first at the University of Chicago for one year and then at Princeton University for another; from 1928 to 1931 he was an assistant professor of mathematics at Cornell University (Davis 1958).

where $p' \equiv dp/dt$. Evans implicitly supposed that t_2 is finite. Thus, from a mathematical standpoint there is no problem in assuming away time discounting because the above integral of (undiscounted) profits is well defined in a finite interval of time.

After the publication of his book in 1930 and after the contributions of his student Charles Roos to economic dynamics in which time discounting was introduced (see below), Evans (1931, 1934) followed the same approach as before and assumed that the objective function that firms maximize is the integral of undiscounted profits over a finite interval of time. Two peculiarities are relevant: the 1934 article was published in an economics journal (in contrast to all previously cited, which were published in mathematical journals) and Henry Schultz, at the University of Chicago department of economics, discussed the 1931 piece. In assessing Evans's theory in economic dynamics Schultz did refer to Roos (1930) (in which firms maximize the integral of discounted profits) but did not mention whether or not it is more appropriate to include a discount factor in the firm's maximization problem. Thus, we can infer that time discounting was a practice that was interpreted differently by members of this community of scientists working on economic dynamics, and it was not understood to be a mathematical necessity.

In his very first published article, Roos (1925) followed Evans's (1924) approach of a monopoly and extended it to the case of a duopoly. He considered two problems of competition among two firms: (1) given the price at the initial period, firm 1 chooses a quantity to be supplied to the marked that maximizes its integral of undiscounted profits over the interval (t_1, t_2) regarding the quantity supplied by firm 2 as not subject to variation, and at the same time firm 2 chooses a quantity to be supplied that maximizes its integral of profits over the same interval regarding the quantity supplied by firm 1 as not subject to variation; (2) given the price level at the initial and end periods, t_1 and t_2, firms choose prices as a function of time that maximizes its own integral of profits considering that only its quantity supplied varies with the price chosen. As pointed out by Roos (1925, 164), both of these are problems that "do not seem to reduce strictly to problems in the calculus of variations, but solutions, however, are given" in the article. Here Roos assumed, as Evans, not only that the cost functions of the firms are quadratic and that the demand function they face is linear in the level and the rate of change of prices, but also that firms maximize an integral of undiscounted profits over an interval—again implicitly assumed to be finite. In the last sections of the article Roos extended his analysis to the case of several producers and the case of, using Evans's terms, the "integral demand law" that Evans (1925) had already referred to.

In a set of future publications we observe that Roos moved from problems with no discounting to those in which future profits are discounted. While he was a national research fellow in mathematics at the University of Chicago, Roos published an article in 1927 in an important economics journal, the *Journal of Political Economy*, in which he extended his previous analysis and that of Evans (1924, 1925): he discussed "the phenomena of competition, monopoly, and cooperation for general functions of demand and cost" (Roos 1927, 635); throughout the paper he assumed, as before, that firms maximize an integral of undiscounted profits over a finite interval of time. It is worth pointing out that Roos had at the time contact with economists as Henry Schultz whom he thanks in the paper for supplying him much of the bibliography and for patiently discussing with him much of the paper.

However, two years later, while discussing issues on business forecasting a few months prior the Great Depression—as he mentions in the introduction, the paper was stimulated by the concerns of bankers and businessmen on whether or not the prosperity observed in the second half of the 1920s was going to continue and if it was "possible to scientifically forecast business conditions" (Roos 1929, 186)—, Roos (1929, 187) considered that firms maximize an integral of discounted profits over a finite interval of time without discussing why discounting was necessary: "each producer attempts to maximize his net profit over an interval of time t_1 to t_2 discounted to the time t_1" (subject to a demand function):

$$\underset{u(t),p(t)}{Max} \quad \Pi = \int_{t_1}^{t_2} [p \cdot u - Q(u, u', p, p', t)] \cdot E(t_1, t) \cdot dt$$

where primes denote derivatives with respect to time; u and p are "the rate of production" (quantity supplied) and the price level, respectively; $Q(\circ)$ is the cost function; and "$E(t_1, t)$ is the discount factor $\exp\left\{-\int_{t_1}^{t} \delta(r) \cdot dr\right\}$ and $\delta(r)$ is the force of interest" (Roos 1929, 188).[12]

In another article published in the *Journal of Political Economy* in 1930 Roos systematized his dynamic approach to economics, in which firms maximize an integral of discounted profits subject to a linear demand function that depends on the level and the rate of change of prices. An interesting aspect of this essay is that Roos (1930) tried to connect the theory of business cycle that

[12] In presenting again such discount factor, Roos (1930, 504) defines the "force of interest" as "the rate of increase of an invested sum S divided by S" and cites a book by the mathematician Lloyd Leroy Smail of 1925 titled *Mathematics of Finance*.

existed at the time—which according to him collected statistical observations, not all consistent with one another, about the price level oscillation, its periodicity and characteristic phases—with a dynamic economic theory that was in fact the mathematical theory that he had been developing over the years. According to this theory changes of parameters could generate oscillations around a trend; thus the cycle was not a consequence of either sunspots or irrational behavior of individuals. Despite being an article published in an economics journal, he again did not explain why a discount factor is necessary or economically meaningful.

Roos (1930) started discussing the simpler case of monopoly but introducing new elements when compared to his previous analysis of the problem: he considered that the goods produced and demanded at time t may not be equal or, what amounts to the same thing, that a part of the current production "will go into stocks on hand" (Roos 1930, 503). Therefore he presented a general model that encapsulated Evans's analysis as a particular case: one in which there is no discounting, the cost function is quadratic and supply and demand are equal at every period. He concluded that "these restrictive assumptions prevented [Evans] from obtaining solutions typical of our economy" (505). One interesting case analyzed by Roos (1930) is a periodic solution that is obtained in the case of no discounting and a general cost function.

In the book he published in 1934 as the first Cowles Commission monograph, Roos (1934) gave a clear economic explanation for assuming that firms discount future profits.[13] In fact, the first time that a discount factor was introduced in the book was in chapter nine when he considered that firms face several risks (from "destruction of plant and equipment by fire or storm" to "technological improvements and new discoveries" (156)) that should be accounted as costs. However:

> Since a remote risk is less important than an immediate one (conditions may change to eliminate the risk and, also, return of capital occurs), to obtain the present (time 0) value of a future (time t) risk it is justifiable to multiply [the total risk] by a discount factor...
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> Roos 1934, 157

[13] As Gerard Debreu argued (cited in Dimand and Veloce 2007, 520), Roos's monograph "is not a genuine Cowles product ... since it was completed by the time its author ... joined the Commission and became its research director in September 1934." Dimand and Veloce (2007, 528) stated that Roos's book consisted "largely [of] papers collected at [Harold] Davis's suggestion."

A few pages later in this same chapter, with arguments similar to those of the time preference theory, Roos (1934, 160) explained with economic arguments why future profits should be discounted in the firm's objective function (the integral of profits over a time interval):

> It is almost universally true that producers prefer early profits to remote or deferred ones. Waiting is an element of cost as truly as effort is, and it should be taken into account. This does not mean, however, that a producer is unwilling to forego present profits in order to obtain greater ones in the future, but it does mean that the expectation of future profits will have to be greater than actual present ones.

In summary, the mathematicians working on economics dynamics in the 1920s and 1930s differed in their use of time discounting. While Griffith Evans kept formalizing the objective function of a firm as the integral of undiscounted profits over a finite interval of time, his student Charles Roos moved from an integral of undiscounted to discounted profits over such interval. Initially Roos did not explain why assuming a discount factor was a sensible hypothesis, but in his 1934 book he provided a clear explanation in economic terms for this strategy. However, contrary to the Cambridge tradition, in Evans's and Roos's models time discounting was an issue related to the behavior of the firms, not of the consumers: both assumed given demand functions that the firms face and did not derived it from any sort of utility maximization problem in which discounting future utilities would appear—Evans was dismissive of the subjective theory of value, as argued by Weintraub (2002, chap. 2), and Roos followed closely his professor in avoiding behavioral assumption about the consumers.[14] By considering firms that each maximize its own integral of profits over a finite time interval, a discount factor could or could not be used by the mathematicians without acquiring a similar ethical dimension as that of Ramsey's intergenerational utility maximization problem.

2.2. The economists: Hart and Tintner

A few economists received well the mathematical approach proposed by Evans and Roos to solve some economic problems: Albert Gailord Hart and Gerhard Tintner. The former became

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[14] While Evans was dismissive of the utilitarian theory of value, Roos was not much in favor of general-equilibrium theory. As Philip Mirowski and D. Wade Hands (1998, 274) point out, although Roos was the first research director of the Cowles Commission, an institution that became in the 1940s the "standard bearer for general-equilibrium theory," and was an "avid supporter of mathematical economics," he was not "enamored of Walras or Pareto."

known for his discussions about the implications of uncertainty to businessmen and mainly policymakers, with an important book published in 1940 based on his Ph.D. dissertation (Hart [1940] 1951). Hart (1937, 273) explained that "even before the onset of the great depression, there was a strong feeling among economists that the chief problem before them was that of business fluctuations [(as it is clear in Roos 1929)]; and this feeling has been intensified by the experience of the last few years." The latter was an enthusiast of the work of Evans and Roos and further developed their approach to economic dynamics, among other things (Fox 2008). In contrast to Hart, Tintner did employ extensively calculus of variations to the problems he studied.

Albert G. Hart received his BA from Harvard in 1930 and his Ph.D. in economics from the University of Chicago in 1936, under Henry Schultz, Frank H. Knight, Jacob Viner, and Theodore O. Yntema (the latter was then a professor of statistics at that university).[15] A decade after obtaining his Ph.D. he became a professor in economics at Columbia University where he stayed until his retirement in 1979 (Earl 2008).[16]

The issue of anticipations became central to Hart's analysis of fluctuations, in which there was a crucial difference between risk (when the holding of anticipations constitute a probability distribution with known parameters) and uncertainty (when the parameters of the distribution of the holding of anticipations are themselves not single valued, i.e. each one has a probability distribution). He then "urges that theorists concentrate their attention on uncertainty rather than on risk" (Hart 1942, 110) and advocates a dynamic approach to economics as the econometric work that Tinbergen and the Cowles Commission were developing (Hart 1945, 544).

Hart plays a central role in the present narrative because he stressed "the time elements of the firm's planning, with special emphasis on anticipations" (Hart [1940] 1951, xi) and brought anticipations of change and uncertainty to bear on business cycle problems. As a consequence, he made time discounting a central piece of his theory of firm's behavior and cycles, as he clearly presented in his 1940 book. In it, he combined in his analysis different strands of understanding of a discount factor.

Hart ([1940] 1951, v) explained in the 1951 preface to his book, "as the events of the 1930's made fluctuations our chief focus of interest, a number of Anglo-American economists (as

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[15] From the acknowledgements in Hart's Ph.D. dissertation, in the article he published based on it (Hart 1937), and in the preface of his 1940 book (Hart [1940] 1951, vi n. 8) we can not know exactly who from the four members of his committee was his advisor, in case there was just one.

[16] Hart's obituary in the Columbia University Record, vol. 23, no. 5 (October 3, 1997), registers that he was a visiting professor in 1946, becoming a professor at the department of economics in 1947 and that he retired in 1977.

well as the Swedes who had pioneered this field), found themselves forced to reformulate their mircro-economics in terms of anticipations." His monograph was part, according to himself, of "the rather substantial literature" on business fluctuations and dynamic problems, and was shaped by Frank W. Taussig's ([1911] 2007, chap. 52) concept of discounted marginal productivity that he had learned as an undergraduate student. The genesis of Hart's book is so important to show how he familiarized himself with those different understandings of time discounting that it is worth quoting the preface to the 1951 edition of his book at length:

> The problem [of anticipations] became acute for me in the winter of 1930-31, in Vienna, when I was forced to think when I was forced to think hard about the questions raised in Hayek's 1931 London lectures—which were published as *Prices and Production*. I had been indoctrinated with the anticipations standpoint in my undergraduate days by F. W. Taussig's insistence that the lag between input and corresponding output made it necessary to handle imputation problems in terms of *discounted* marginal productivity.[17] This approach was still more natural by the concentration of the "trio seminar" of Haberler, Hayek and Morgenstern in Vienna in the autumn of 1930 upon Fisher's *Theory of Interest*. The insistence of the students in the seminar on reformulating Fisher in terms of Böhm-Bawerk (!) gave plenty of exercise in input-output lags. When confronted with Hayek's business cycle theory, I found I could not be either for it or against it without new constructions. …
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> These questions were a focus of discussion among the graduate students at Chicago in 1931-34, and not unnaturally led me to a dissertation topic. … In the autumn of 1934, when I arrived at the London School of Economics with a rough draft under my arm (written just after landing in England), I found a lively discussion in progress.[18] J. R. Hicks was just preparing the lectures which became his article on "Wages and Interest—the Dynamic Problem." Shortly, there was circulated a mimeograph from Erik Lindahl, anticipating a key section of his later *Money and Capital*. In this atmosphere, I rounded off in December 1934 my first

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[17] Hart (1937, 273 n. 1) had already pointed out that Taussig's concept of "'discounted marginal productivity' led [the former] to a study of anticipations."

[18] In the 1940 preface to the book Hart [1940] 1951, xii) thanks the University of Chicago "for a generous grant of leave in the academic year 1934-35 which enabled him greatly to extend his contacts with economists interested in these questions and which provided leisure for a first attempt to analyze them at length."

> reasonably complete draft of the material which went into this monograph. After various vicissitudes, my dissertation was accepted at the University of Chicago in the spring of 1936. Various offshoots of it got into print during 1936-37. This monograph, published in 1940, was a revision of the parts of the dissertation dealing most specifically with the firm.[19]
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> Hart [1940] 1951, v-vii

From Hart's account of his book a decade after its first edition, it is clear that Hart was well acquainted with the work of a variety of economists that was connected to his own enterprise—other relevant names cited in the book and not mentioned in the passage above are Gunnar Myrdal, George L. S. Shackle, Frank Knight, Jacob Marschak, and Gerhard Tintner. In all these works the idea that something at different points of time should be treated as different things was clearly understood. Moreover, Hart ([1940] 1951, v n. 1) claimed that, in the United States, Roos "published in 1925-1934 a series of writings which he asserts 'pioneered and abandoned' the expectational approach" to which Hart was contributing.[20] However, he seems not to cite Roos's works in the chapters of his 1940 book (and in his 1937 article, for that matter).

In this literature, the appropriate objective function that firms maximize (if discounted or undiscounted flow of profits or any other variable) was an open question. Hart (1937, 278 n. 5) justified his use of discounted profits (discounted "net receipts") as the firm's objective "as the most natural extension of the traditional 'maximum net receipts'" and that it has been "used by several of the writers in English on anticipations" such as Harold Hotelling (1931) and John R. Hicks (1935).[21] However, continued Hart, Griffith Evans (1930) and Kenneth Boulding (1935) proposed that firms maximize "undiscounted net receipts" and the "average rate of return on

[19] Hart's 1937 article was also based on his dissertation.

[20] Hart ([1940] 1951, v n.1) cited Roos's "most accessible of his early papers": those published in the *Journal of Political Economy* in 1927 and 1930, and noted that Roos "described the firm as maximizing discounted profits" (as discussed in section 2.1). Later in the preface Hart cites Samuelson's literature review on dynamic economics to the American Economic Association (Samuelson 1948) which was "an essay on the logic and applications of differential equations, with no discussion of anticipations anywhere visible" (viii).

[21] Hotelling (1931) not only modeled firms as maximizers of the present value of their profits over time but also discounted future utilities at a constant rate: he was one of the first to cite Ramsey's 1930 article (Duarte Forthcoming(b)) but, differing from the latter, he applied a discount factor to future utilities even in the case of integrating them over a finite time interval (Hotelling 1931, 143). See also Hotelling's argument against the criticism of discounting future utilities for the case of mining he analyzed (145-146).

capital over the life of the enterprise," respectively.[22] Boulding (1942) himself noted that time discounting was not a common practice among economists working with the theory of the firm. In his review of the literature of the 1930s to the early 1940s he refers to Hart, Hicks, Hotelling, and Tintner, but Evans or Roos are to be found nowhere.

Therefore, Hart's contributions to anticipations and the business cycle are important for bringing together a diverse set of works on economics dealing with these issues, including, to a lesser extent, Evans and Roos. His discussions on the use of a discount factor to the firm's objective function shows that a diverse understanding about this practice existed up to the 1940s. Tintner not only contributed to this literature but, as Hotelling, also considered discounting future utilities and employed heavily calculus of variations to analyze intertemporal problems related to monopoly, distribution of income, business fluctuations, utility maximization and choice theory.

Gerhard Tintner, born in Germany in 1907 and educated in economics, statistics and law in Vienna (he obtained his doctorate in 1929 at the University of Vienna), became in the mid-1930s an enthusiast of the work of Evans and Roos on economic dynamics.[23] In contrast to Hart, as already noted, here we see him explicitly using calculus of variations and extending Evans's and Roos's works.[24] Despite Tintner's use of a time discount factor in many problems, going further than Hart when he employed it to the utilities of individuals at different points in time (but not in the same way that modern growth economists do), we observe a similar ambiguity with respect to such use as that of Roos.

From the mathematical notation to the problems analyzed to the references cited and to the mathematical tools, it is clear from all this that in the set of papers that Tintner published from the mid-1930s to the 1940s, most of them in *Econometrica*, he followed Evans and Roos closely.[25] For instance, in order to discuss income distribution over time Tintner (1936(a)) applied to the utility

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[22] Boulding (1942, 793) explained that the understanding shared by those assuming that firms maximize the discounted "net revenue" was that discounting was necessary to take into account the opportunity cost of investment: "The discounting presumably is to be done for each period of time at that rate of interest which represents the alternative cost of employing capital in the occupation in question; that is, at the rate which the entrepreneur could obtain in other investments."

[23] In the early 1930s Tintner was in the *Institut fuer Konjunkturforschung*, in Vienna. I do not know whether or not he met Hart there, in the winter of 1930-31. In 1936 he accepted an invitation to become a fellow of the Cowles Commission—Roos was then the research director of Cowles—, but resigned less than a year thereafter, in September 1937, to join the faculty of Iowa State College (Hart eventually joined the faculty of this college and became Tintner's colleague).

[24] Tintner's enthusiasm with the work of Roos appears also in his favorable review of Roos's 1934 book for the *Journal of Political Economy* in 1936.

[25] The same applies to Tintner's (1942(b)) attempt to discuss business fluctuations in a simple linear model that echoes both Evans (1931) and Roos (1930), but much more the former than the latter.

function the same idea that Evans and Roos used to the demand function faced by the firms: he assumed that the utility function depended not only on the quantities of the n goods of the economy, "but also on their flow in time" (i.e., their derivatives with respect to time). In this article we can only infer that he was not using a discount factor from the budget constraint of the intertemporal utility-maximization problem (p. 63), because he did not write explicitly what was the intertemporal utility function (he just stated that it was a function of the quantity and the change over time of the goods available).

From the other related papers it is clear that Tintner wished to relate the dynamic theory of the firm expounded by Evans and Roos to a dynamic theory of choice—if the utility function depended on expected prices, which in turn depend on price tendencies, the time derivatives of prices would be arguments both of the utility function (which he called "dynamic utility function") and the demand function of the individuals and, thus, monopoly firms would maximize profits subject to a demand function that is a differential equation, as in Evans's and Roos's works. In such integrated framework issues on "expectations, economic horizon, and the role of time in economic life in general" would play a significant function (Tintner 1937, 161). However, only in the paper published in 1942 that Tintner would turn his attention to a dynamic theory of choice.

With respect to the use of a discount factor, Tintner incorporated it only in 1938. In his 1937 essay he stated, as Evans, that monopoly firms maximize the undiscounted integral of profits over an implicitly assumed finite interval of time (the notation is exactly the same as that of Evans or Roos):

$$\underset{p(t)}{Max} \quad \Pi = \int_{t_0}^{t_1} \pi(p, p') \cdot dt$$

In the three papers published in 1938 and 1939 Tintner adopted time discounting in the intertemporal utility maximization and demand problems he studied. As in the 1936 piece (Tintner 1936(a)), the use of a discount factor can be inferred only from the budget constraint or the expenditure function, since Tintner simply assumed that the utility function depends on quantities which the individual expects to consume over time (in other words, he did not write the intertemporal utility function as the integral over time of instantaneous utility functions—discounted or not). Tintner closed his 1939 article relating his discussion about the influence of income, prices and interest rates upon expenditures at different points in time to that of Irving Fisher and the theory of time preference:

> The difference between our approach and the theory of Fisher is the following: We assume explicitly that the utility function or indifference map of the individual depends not merely upon the undiscounted expenditure stream … (Fisher's income stream), but upon all the quantities of all commodities which the individual plans to consume at all points in time … . The time preference, if any, should be expressed in the form of the utility function.
>
> Tintner 1939, 270

In this passage it is clear that Tintner, as Roos, was reluctant to build his "dynamic utility function" upon explicit behavioral assumptions derived from a theory of time preference. This led him to discuss his dynamic theory of choice under uncertainty—here understood exactly in the way of Hart, who was then his colleague at Iowa State College—, in which time discounting was important but not a salient feature of his analysis (Tintner 1942(a), (c), (d)). This set of papers constituted the last step towards his dynamic approach to economics in which expectations, time, and anticipations were central components.

In conclusion, in this section I analyzed the work of mathematicians and economists who were among the first to apply calculus of variations to economic problems. These scientists understood that the technique developed by Euler and Lagrange in the eighteenth century—which "deals with problems of finding a function or a path that maximizes some criterion" (Kamien 2008)—was applicable to many problems in economics involving optimal decisions through time, whose solution was a path of actions rather than a single decision.

While Ramsey (1928) condemned on ethical grounds the use of a discount factor in problems of allocating resources among different generations in a community, the mathematicians and economists working on economic dynamics from the 1920s to the 1940s had a diverse understanding about the use of such factor, but not because of ethical considerations. Even before the Great Depression, they became interested in problems of business fluctuations. By discussing these problems they took stands on the proper objective of a firm and, therefore, on whether future profits ought to be discounted. Griffith Evans postulated that firms maximize the undiscounted integral of profits over a finite interval of time. Charles Roos and Gerhard Tintner adopted time discounting after some point, but in their first papers in this literature firms maximize the undiscounted integral of profits over time. Albert Hart, in contrast, saw his postulate that firms maximize the integral of discounted profits as a natural extension to economic dynamics of the static principle of profit maximization. In any circumstances, Hart ([1940] 1951) and Boulding (1942) provide evidence that this lack of consensus about time discounting characterized the literature on the dynamic behavior of firms.

From all these names, Tintner was the only one to extend his dynamic analysis to the behavior of consumers. Nonetheless, the ethical problem raised by Ramsey did not turn up in his analysis: Tintner avoided writing his intertemporal utility function as an integral over time of instantaneous utility functions, as Ramsey did; he just postulated the existence of a utility function that had as arguments the quantities which the individual expects to consume over time (this includes Ramsey's integral of instantaneous utility functions as a special case). Therefore, he made no explicit statement about discounting future utilities, keeping away ethical judgments on the value of the utility of future generations relative to that of the living generation.

The question of discounting future utilities, be that of an individual or of a generation, was addressed by economists working on optimal growth theory in the 1960s—who employed optimal control and dynamic programming methods, which are generalizations of the calculus of variations. A clear axiomatization of time preferences and the consolidation of representative agent models (together with the increasing use of recursive methods in economics) set the stage for turning discounting future utilities into a rather technical question, gradually deprived of an ethical content.

3. A technically indispensable practice[26]

In their entry to the *New Palgrave* Christopher Chabris, David Laibson and Jonathon Schultz (2008) provide an overview of the recent literature on intertemporal choices and on time discounting. They state that "the theory of discounted utility is the most widely used framework for analyzing intertemporal choices" both in descriptive (positive economics) and in prescriptive (normative economics) terms. They continue: "descriptive discounting models capture the property that most economic agents prefer current rewards to delayed rewards of similar magnitude. … Normative intertemporal choice models divide into two approaches. The first approach accepts discounting as a valid normative construct, using revealed preference as a guiding principle. The second approach asserts that discounting is a normative mistake … [and] adopts zero discounting … as the normative benchmark." They then assert that the "most widely used discounting model" is that of Ramsey (1928). Although it is true, as discussed in section 1, that Ramsey (1928) did consider the case of discounting future utilities, he is mostly remembered nowadays for condemning the discounting of utilities of future generations.[27]

What the authors most probably had in mind is the Ramsey-Cass-Koopmans growth model (also known as the "Ramsey model"), in which there is a representative agent that discounts future utilities at a constant rate.[28] Nevertheless, they highlight the different conceptions of time discounting: as a description of individuals' behavior and as a device of theories of socially optimal behavior (in which there is one stream accepting discounting as a valid normative hypothesis and another arguing in favor of no discounting in normative models). This tension between the positive and normative concepts of time discounting played an important role in the unfolding discussion of optimal growth models in the 1960s and the 1970s.

3.1 Axiomatizing Time Preferences

As already noted, a crucial element in the developments of intertemporal models of growing economies is the axiomatization of time preferences done by economists like Tjalling Koopmans (1960, 1964, 1972), Koopmans, Peter Diamond and Richard Williamson (1964), and Diamond (1965). What these authors did was to transform Fisher's concept of impatience (that individuals facing the alternatives of having a given reward today or in the future opt for current reward) into a central element defining an ordinal intertemporal utility function. As Koopmans (1960, 288) argued, it is this function that defines time preferences either of an individual consumer, or of an aggregate of such individuals, or finally choices in a centrally planned economy.

As a consequence of this axiomatization, economists now knew exactly how to do what Ramsey (1928) had done and what Tintner was reluctant to do: to write the lifetime utility function as the integral over time of the instantaneous utility function, and work with it in an Arrow-Debreu general equilibrium framework. Another consequence was that economists had a way of doing

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[26] **This section is <u>incomplete</u> and I just sketch in it the arguments to be developed and the related literature.**

[27] For example, after stating the intertemporal utility function of the Ramsey-Cass-Koopmans model, Blanchard and Fischer (1989, 81-2 n.4) recognized that because they assumed a positive rate of time preference they "depart from Ramsey who, interpreting the maximization problem as the problem solved by a central planner, argued that there was no ethical case for discounting the future."

[28] The discount factor is employed precisely because in this model utility and production functions are assumed not to have any kind of "satiation levels" as Ramsey's bliss. See Blanchard and Fischer (1989, chap. 2), Romer (1996, chap. 2, part A) and Barro and Sala-i-Martin (2003, chap. 2).

what Ramsey did not do explicitly: to write the lifetime utility function as the integral of discounted instantaneous utilities:

$$\underset{c_t,k_t}{Max} \quad U_0 = \int_0^{\infty} e^{-\theta \cdot t} u(c_t) \cdot dt \qquad s.t.: \begin{cases} f(k_t) = c_t + \dfrac{dk_t}{dt} + (\delta + n) \cdot k_t \\ k_t, c_t \geq 0 \quad \forall t; k_0 \quad given \end{cases}$$

where c_t and k_t are per capita consumption and capital stock, n and δ are the rates of population growth and of depreciation, respectively. This problem is that of a benevolent social planner who, at time zero, maximizes social welfare (which is the lifetime utility of the representative household). It is analogous to Ramsey's problem presented in section 1, with the substantial difference that the use of a discount factor allowed economists not to use Ramsey's trick of the existence of a bliss.

Although the narrative of economic dynamics (and of optimal economic growth) can be seen as a struggle to find ways of dealing with time preferences, the axiomatization proposed by Koopmans and others did not mean that a consensus would soon emerge in the growth literature. Alternative concepts of convergence were developed to guarantee that maximization of intertemporal utility was a well-defined mathematical problem, and the discussion on a discount rate for a society generated heat and light.

3.2 The Growth Literature, Recursiveness and Discussions on Discount Rates

The effort economists made for constructing a behavioral basis of an ordinal intertemporal utility function that characterized either the behavior of an individual or of a social planner did not prevent serious criticisms to employing a social discount factor. In the 1960s discounting utilities of future generations was a controversial issue. See, for example, Dobb (1960, chap. 2) and more specifically Marglin (1963), Lind (1964), Tullock (1964), and Sen (1967).

Another important development in economics after World War II relevant for understanding the stabilization of time discounting was the dynamic programming methods advanced by Richard Bellman (1953, 1957), David Blackwell (1965) and others. These methods provided recursive solutions to dynamic problems that were easier to obtain relative to the standard calculus of variations approach. Among the many situations studied in the dynamic programming literature, there is the special case of maximizing a payoff (objective) function over the infinite future, subject to some constraints. The infinite horizon brings to economists the possibility of obtaining a time-invariant functional solution: it is easily obtainable when one explores the recursiveness of the problem, i.e., the fact that decisions over the indefinite future are just a decision on an immediate action as a function of the current situation (the "state" of the economy) in a problem whose structure recurs each period (Stokey and Lucas, 1989, 5).[29] However, the problem of maximizing over the indefinite future is that a solution may imply infinite payoffs. In order to avoid this problem of the inexistence of a finite solution, time discounting is introduced: under certain conditions, maximizing discounted payoffs has a finite solution, as stated by the contraction mapping theorem[30]—which is "an extremely simple and powerful fixed point theorem" (Stokey and Lucas, 1989, 49).

Before continuing, it is worth quoting Bellman's and Blackwell's view on time discounting. In discussing the problem of a firm that minimizes costs from today to the indefinite future, Bellman (1957, 156) wrote:

> If we wish to consider an unbounded period of time over which this [cost minimization] process operates, we must introduce some device to prevent infinite costs from entering.
>
> The most natural such device is that of discounting the future costs, using a fixed discount ratio … for each period. This possesses a certain amount of economic justification and a great deal of mathematical virtue, particularly in its invariant aspect.

In discussing cost minimization, Bellman does not refer to the earlier literature on maximizing profits over time.

Blackwell (1965, 226), in his turn, treated time discounting from a purely technical perspective:

> This total reward may well be infinite… .We shall avoid this problem by introducing a discount factor β, $0 \leq \beta < 1$, so that unit reward on the *n*th day is worth only β^{n-1}, and shall try to maximize the total expected reward over the infinite future.

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[29] Bellman (1957, 230) referred to this time invariance introduced of the functional solution by the infinite horizon as "homogeneity": "The infinite problem is, as usual, simpler than the finite case because of the homogeneity introduced by infinite time; after any initial actions, we are confronted by a problem of the same type, with different initial values." The same idea was present in the optimal growth literature as summarized by Koopmans (1967, 2).

[30] See Stokey and Lucas (1989, 49-55).

When we look at the growth literature developed in the 1970s—the one in which agents maximize utility to decide how much to consume or save; therefore, it is not the literature of the Solow model—we see economists considering versions of their models with and without discounting.[31] At the same time, there were economists proposing alternative optimality criteria, as Boyer (1975), for instance. Another important and well known criterion at the time was the so-called "overtaking criterion" proposed by von Weizsäcker (1965): a consumption path, $\{c_1(t)\}_0^\infty$, is better than another path, $\{c_2(t)\}_0^\infty$, if there exists an instant T such that the (undiscounted) lifetime utility of the first path is greater than that of the latter path for every point in time after that T (even if the utility of the path $c_1(t)$ is lower than that of $c_2(t)$ before T, the former is optimal if it overtakes the latter in terms of lifetime utility after T):

$$\int_0^t u(c_1(\tau))\cdot d\tau > \int_0^t u(c_2(\tau))\cdot d\tau \qquad \forall t \geq T$$

Note that in order to choose the best among two alternative consumption paths one needs to compute integrals for finite horizons. Therefore, the problem of an unbounded integral $\int_0^\infty u(c_1(t))\cdot dt$ can no longer exist. However, in contrast to the strategy of maximizing the discounted lifetime utility (or minimizing the lifetime distance of the utility to the bliss level), the overtaking criterion does not choose between every conceivable pair of paths, as pointed out by Koopmans (1967, 5). Nonetheless, he continues, "the partial ordering defined by [this] criterion suffices for determining a unique optimal path in the circumstances assumed by Ramsey [(1928)]," which are basically the same as those of the Ramsey-Cass-Koopmans model.

This overtaking criterion was used by economists who did not want to employ a discount factor in their models, or by those who wishes to compare the results with discounting to those without it. Over time, however, economists increasingly adopted one way of modeling intertemporal growth problems, in which time discounting became a technical requirement. Clearly by the late 1980s, this was the predominant view, as exemplified by the books by Stokey and Lucas (1989) and Blanchard and Fischer (1989).[32]

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[31] See Brock and Mirman (1973) and Brock (1973).

[32] Not many years before these books were published, Sargent (1987, chap. 1) presented dynamic programming in general terms and applied it to the Ramsey-Cass-Koopmans model without discussing the appropriateness of

4. Concluding remarks

This is a narrative in which a technical aspect of economic models had its meaning and use negotiated among different economists (and mathematicians) over time. Time discounting was controversial not only because it acquired an ethical dimension when applied to future utilities of generations. Even the mathematicians and economists working on the dynamic behavior of firms understood the necessity of a discount factor in different ways—such different understandings do not arise from an ignorance of ideas about preferences over time, as impatience.

In the 1960s the optimal growth literature that emerged postulated individuals (or a planner) maximizing an objective function subject to restrictions (in contrast to the earlier growth literature that assumed *ad hoc* consumption or saving functions). In it, there were alternative criteria of convergence that allowed economists to specify variants of their model with and without time discounting, despite the contributions of other economists that found explicit behavioral basis for a discounted intertemporal utility functions. These contributions, together with the development of dynamic programming methods paved the way for the use of a time discount rate to become the most widely used framework for analyzing intertemporal choices, to use the words of Chabris, Laibson and Schultz (2008).

Over time, with the spread of the representative agent growth model known as the Ramsey-Cass-Koopmans model, discounting the utilities of future generations had less and less an ethical dimension to it. Economists working on economic growth (and on intertemporal macroeconomics models more generally) increasingly treated the use of a discount factor as a technical requirement that guarantee the existence of finite solutions, which are easy to obtain with recursive techniques. From an ethically indefensible to a technically indispensable practice, there is a path—clearly one not straightforward—leading economists through the wild understandings about the use of a time discount factor.

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discounting future utilities. In appendix A.7 he briefly discussed the "discounted dynamic programming" and referred to the forthcoming book by Stokey and Lucas, so that discounting was also a technical issue.

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